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Theta Constants and Teichmüller Modular Forms

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In this paper, we determine a primitive Teichmüller modular form of degree $g \geqslant 3$ over $\mathbf{Z}$ obtained from dividing the product of even theta constants by a certain

1. INTRODUCTION

Let g be a positive integer $\geqslant 3$. The problem treated in this paper is raised from calculating the expansion of a special Teichmüller modular form of degree g, a root of the product of even theta constants of degree g, by local coordinates on the moduli space of curves of genus g induced from Mumford's uniformization theory. Teichmüller modular forms of degree g are defined as global sections of the automorphic line bundles on the moduli space of smooth and proper curves of genus g (cf. [6]). Besides examples obtained from Siegel modular forms by the pull back of the Torelli map τ, for each g, there exists an interesting Teichmüller modular form of degree g obtained by Tsuyumine [16] as a root of the product $\theta_g(Z)$ ($Z \in$ the Siegel upper half space of degree g) of theta constants

$$\theta_g \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} (Z) = \sum_{\mathbf{n} \in \mathbf{Z}^g} \exp[\pi \sqrt{-1}\,(\mathbf{n}+\mathbf{a})\, Z^t(\mathbf{n}+\mathbf{a}) + 2\pi \sqrt{-1}\,(\mathbf{n}+\mathbf{a})^t\, \mathbf{b}]$$

with characteristic $\mathbf{a}, \mathbf{b} \in \{0, 1/2\}^g$ such that $4\mathbf{a}^t\mathbf{b}$ is even. This form, which multiplied by a certain number can be defined over $\mathbf{Q}$, is not induced from Siegel modular forms when $g = 3$ by the weight argument, but the author does not know whether the situation holds for $g \geqslant 4$. As is shown in [6], using Mumford's uniformization theory [11], we have a canonical ring

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homomorphism κ from the ring of Teichmüller modular forms of degree g over $\mathbf{Q}$ into $A \hat{\otimes} \mathbf{Q}$, where A is the ring of formal power series of $y_1, ..., y_g$ over

$$R = \mathbf{Z}\left[x_k, \prod_{i \neq j} \frac{1}{x_i - x_j} (i, j, k \in \{\pm 1, ..., \pm g\}) \right] \qquad (x_k\text{: variables}).$$

Moreover, it is shown in [6] that for the Teichmüller modular form $\tau^*(\varphi)$ induced from a Siegel modular form φ of degree g over $\mathbf{Q}$,

$$\kappa(\tau^*(\varphi)) = F(\varphi)|_{q_{ij} = p_{ij}},$$

where

$$F(\varphi) \in \mathbf{Q}[q_{ij}^{\pm 1}(i, j \in \{1, ..., g\}, i \neq j)][[q_{11}, ..., q_{gg}]]$$

(q_{ij}: variables with symmetry $q_{ij} = q_{ji}$) denotes the Fourier expansion of φ and $p_{ij} \in A$ $(i, j = 1, ..., g)$ denote the universal periods defined in [5]. Since $F(\theta_g)$ has integral Fourier coefficients, there exists a unique integer N_g such that $\kappa(\tau^*(\theta_g))/N_g = (F(\theta_g)/N_g)|_{q_{ij} = p_{ij}}$ is in A and primitive (i.e., $\not\equiv 0$ modulo p for any prime p), and that $\tau^*(\theta_g)/N_g$ times a certain positive integer has a root as a Teichmüller modular form of degree g over $\mathbf{Q}$.

In this paper, first we determine this N_g as

$$N_g = \begin{cases} -2^{28} & (g = 3) \\ 2^{2^{g-1}(2^g - 1)} & (g \geqslant 4), \end{cases}$$

and we show that $\tau^*(\theta_g)/N_g$ itself has a root f_g as a Teichmüller modular form over $\mathbf{Q}$ by calculating minimal degree terms of $F(\theta_g)$ with respect to $q_{11}, ..., q_{gg}$. We note that the Siegel modular form θ_g/N_g can be defined over $\mathbf{Z}$ which is not a trivial fact because there are Siegel modular forms of degree $g \geqslant 4$ vanishing on the Jacobian locus. Second, from this calculation and the existence of a root of $\kappa(\tau^*(\theta_g))/N_g = \kappa(f_g)^2$ in $A \hat{\otimes} \mathbf{Q}$, we have

THEOREM. *Let $g \geqslant 3$ be an integer, let $X_{ij} (i, j \in \{1, ..., g\}, i \neq j)$ be variables with $X_{ij} = X_{ji}$, and let $\psi : \mathbf{C}[X_{ij}^2] \to R \otimes \mathbf{C}$ be the ring homomorphism over $\mathbf{C}$ satisfying that*

$$\psi(X_{ij}^2) = \frac{(x_i - x_{-j})(x_{-i} - x_j)}{(x_i - x_j)(x_{-i} - x_{-j})}.$$

For all $\mathbf{b} = (b_k)_{1 \leqslant k \leqslant g} \in \{0, 1/2\}^g$ with $|\mathbf{b}| = \sum_{k=1}^{g} |b_k| \in \mathbf{Z}$, put

$$P_{\mathbf{b}} = \tfrac{1}{2} \sum_{S \subset \{1, ..., g\}} (-1)^{2\mathbf{v}_S^t \mathbf{b}} \prod_{i \in S,\, j \notin S} X_{ij},$$

where $\mathbf{v}_s = (v_k)_{1 \leqslant k \leqslant g}$ *is given by* $v_k = 1 (k \in S)$, $v_k = 0 (k \notin S)$, *and let* $P \in \mathbf{Z}[X_{ij}^2]$ *be their product. Then* P *has no root in* $\mathbf{C}[X_{ij}]$, *but* $\psi(P)$ *has a root in* R.

It is not so easy to calculate $\sqrt{\psi(P)}$ even when $g = 3$ and seems to be interesting because we can conclude that f_g is not induced from Siegel modular forms if $\sqrt{\psi(P)} \notin \mathrm{Im}(\psi)$ (which can be checked when $g = 3$).

Before closing the introduction, we shall review some results in [8] concerning the above f_g. In [8], constructing a smooth and proper curve over $A[\prod_{i=1}^{g} 1/y_i]$ uniformized by a universal Schottky group, we show that the above κ can be defined for Teichmüller modular forms with coefficients in any module M. This and the irreducibility of the moduli space of curves [2] imply that f_g is, in fact, defined over $\mathbf{Z}$ and primitive as a Teichmüller modular form, and that $(F(\theta_g)/N_g)|_{q_{ij} = p_{ij}}$ has a root in A. Moreover, it is shown in [8] (see also [7]) that f_3 is obtained from Mumford's isomorphism [13, Theorem 5.10] and that the ring of Teichmüller modular forms of degree 3 over $\mathbf{Z}$ is generated by that of Siegel modular forms of degree 3 over $\mathbf{Z}$ and by f_3 (Tsuyumine [15] determines the ring structure of Siegel modular forms of degree 3 over $\mathbf{Q}$).

2. REVIEW ON TEICHMÜLLER MODULAR FORMS

2.1. In this section, we mainly review some results on Teichmüller modular forms and their canonical expansion constructed by Mumford's uniformization theory. Throughout this paper, we assume that $g \geqslant 3$.

Let $\mathcal{M}_g$ denote the moduli stack classifying smooth and proper curves of genus g (cf. [2, Section 5]). Let $\pi: \mathscr{C} \to \mathcal{M}_g$ be the universal curve, and let λ be the invertible sheaf $\bigwedge^g \pi_*(\Omega_{\mathscr{C}/\mathcal{M}_g})$ on $\mathcal{M}_g$. For an integer h and a module M, put

$$T_{g,h}(M) = \Gamma(\mathcal{M}_g, \lambda^{\otimes h} \otimes M),$$

and we call these elements Teichmüller modular forms of degree g and weight h with coefficients in M (or defined over M if M is a $\mathbf{Z}$-algebra). Let $\mathscr{X}_g$ denote the moduli stack of principally polarized abelian schemes of relative dimension g, and let μ be the invertible sheaf $\bigwedge^g \rho_*(\Omega_{\mathscr{A}/\mathscr{X}_g}) = \bigwedge^g \varepsilon^*(\Omega_{\mathscr{A}/\mathscr{X}_g})$ on $\mathscr{X}_g$, where $\rho: \mathscr{A} \to \mathscr{X}_g$ denotes the universal abelian scheme with zero section $\varepsilon: \mathscr{X}_g \to \mathscr{A}$. As in [4, Chap. V], Siegel modular forms of degree g and weight h with coefficients in M are defined as elements of

$$S_{g,h}(M) = \Gamma(\mathscr{X}_g, \mu^{\otimes h} \otimes M).$$

Then associating curves to their Jacobians with canonical polarization, we have the Torelli map $\tau: \mathscr{M}_g \to \mathscr{X}_g$ which induces the homomorphism $\tau^*: S_{g,h}(M) \to T_{g,h}(M)$.

We consider a special Siegel modular form of degree g over $\mathbf{C}$ as a holomorphic function on the Siegel upper half space

$$H_g = \{Z \in M_g(\mathbf{C}) \mid {}^tZ = Z, \operatorname{Im}(Z) > 0\}$$

of degree g. Put $h(g) = 2^{g-3}(2^g+1)$. For all $4h(g)$ pairs $(\mathbf{a}, \mathbf{b})$ of row vectors with g entries in $\{0, 1/2\}$ such that $4\mathbf{a}{}^t\mathbf{b}$ is even (in which case $(\mathbf{a}, \mathbf{b})$ is called even), let

$$\theta_g \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} (Z) = \sum_{\mathbf{n} \in \mathbf{Z}^g} \exp[\pi\sqrt{-1}\,(\mathbf{n}+\mathbf{a})\,Z^t(\mathbf{n}+\mathbf{a}) + 2\pi\sqrt{-1}\,(\mathbf{n}+\mathbf{a})^t\,\mathbf{b}]$$

be the theta constants of characteristic $(\mathbf{a}, \mathbf{b})$, and let

$$\theta_g(Z) = \prod_{\mathbf{a},\,\mathbf{b}} \theta_g \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} (Z)$$

be their product. It is well known (cf. [9]) that $\theta_g \in S_{g,\,2h(g)}(\mathbf{C})$, and by [14, Proposition 3] this is seen to be a prime element in the ring of Siegel modular forms of degree g over $\mathbf{C}$. Since θ_g has Fourier expansion $F(\theta_g)$ as a power series of $q_{ij} = \exp(2\pi\sqrt{-1}z_{ij})$ $(Z = (z_{ij})_{1 \leqslant i,\, j \leqslant g} \in H_g)$ with coefficients in $\mathbf{Z}$, by the q-expansion principle (cf. [4, p. 140]), θ_g is defined over $\mathbf{Z}$.

2.2. Let

$$[a, b; c, d] = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

denote the cross ratio of four points. Let $x_{\pm 1}, \ldots, x_{\pm g}, y_1, \ldots, y_g$ be $3g$ variables, and let A be the ring of formal power series of $y_1, \ldots, y_g$ with coefficients in the ring

$$R = \mathbf{Z}\left[x_k, \prod_{i \neq j} \frac{1}{x_i - x_j} \; (i, j, k \in \{\pm 1, \ldots, \pm g\}) \right].$$

Let Φ be the free subgroup of $PGL_2(\Omega)$ (Ω: the quotient field of A) generated by

$$\varphi_k = \begin{pmatrix} x_k & x_{-k} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y_k \end{pmatrix} \begin{pmatrix} x_k & x_{-k} \\ 1 & 1 \end{pmatrix}^{-1} \operatorname{mod}(\Omega^\times),$$

and for each i, $j=1, ..., g$, let $\psi_{ij}\colon \Phi \to \Omega^{\times}$ be the map given by

$$\psi_{ij}(\varphi)=\begin{cases} y_i & (\text{if} \quad i=j,\ \varphi\in\langle\varphi_i\rangle) \\ [x_i, x_{-i}; \varphi(x_j), \varphi(x_{-j})] & (\text{otherwise}) \end{cases}$$

which depends only on double coset classes $\langle\varphi_i\rangle\backslash\varphi/\langle\varphi_j\rangle$. Then it is shown in [5, Section 2] that the infinite product

$$p_{ij}=\prod_{\varphi}\psi_{ij}(\varphi)$$

(φ runs through all representatives of $\langle\varphi_i\rangle\backslash\Phi/\langle\varphi_j\rangle$) is convergent in A and particularly satisfies the congruence

$$p_{ij}\equiv\begin{cases} y_i & \operatorname{mod}(I^2) & \text{for} \quad i=j \\ [x_i, x_{-i}; x_j, x_{-j}] & \operatorname{mod}(I) & \text{for} \quad i\neq j, \end{cases}$$

where I is the ideal of A generated by $y_1, ..., y_g$. Then by results of [12; 4, Chap. III], there exists a semiabelian scheme J_{Φ} over A with multiplicative periods $p_{ij}(i, j=1, ..., g)$. Let χ_k $(k=1, ..., g)$ be the basis of $\operatorname{Hom}(\mathbf{G}_m^g, \mathbf{G}_m)$ over $\mathbf{Z}$ defined by $\chi_k((z_i)_i)=z_k$. Then $\Gamma(J_{\Phi}, \Omega_{J_{\Phi}/A})$ is known to be a free A-module generated by $d\chi_k/\chi_k$ $(k=1, ..., g)$ (cf. [4, Chap. III], and hence this g-fold exterior power is an invertible A-module generated by

$$\omega=(d\chi_1/\chi_1)\wedge\cdots\wedge(d\chi_g/\chi_g).$$

Let K be a nonarchimedean complete valuation field, and let $S_{g/K}$ be the space of all Schottky groups Γ over K with free g generators $\gamma_1, ..., \gamma_g$ which becomes a K-analytic open subspace of

$$\{(\alpha_k, \alpha_{-k}, \beta_k)_{1\leqslant k\leqslant g} \mid \alpha_{\pm k}\in\mathbf{P}^1(K),\ \beta_k\in K^{\times}\}$$

by the relation

$$\gamma_k=\begin{pmatrix} \alpha_k & \alpha_{-k} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \beta_k \end{pmatrix}\begin{pmatrix} \alpha_k & \alpha_{-k} \\ 1 & 1 \end{pmatrix}^{-1} \quad \operatorname{mod}(K^{\times}).$$

Let C_{Γ} be the Mumford curve over K associated with Γ [11]. Then by results of Manin and Drinfeld [10], $p_{ij}|_{x_{\pm k}=\alpha_{\pm k},\ y_k=\beta_k}$ $(i, j=1, ..., g)$ are convergent and become multiplicative periods of the Jacobian variety J_{Γ} of C_{Γ}, and hence ω induces a canonical generator of $\bigwedge^g \Gamma(J_{\Gamma}, \Omega_{J_{\Gamma}/K})$ which we denote by the same symbol. The correspondence $\Gamma\mapsto C_{\Gamma}$ for Schottky groups Γ of rank g over K induces a morphism ξ_K from $S_{g/K}$ to the K-analytic orbifold associated with $\mathscr{M}_g\otimes K$. Therefore, for any $f\in T_{g,h}(K)$,

$\xi_K^*(f)\,\omega^{-h}$ is a K-analytic function on $S_{g/K}$, and hence it is an element of $A[\prod_{i=1}^{g} 1/y_i]\,\hat{\otimes}\,K$.

2.3. Let k be a field of characteristic $\neq 2$, and let $K = k((z))$ be the valuation field over k with prime element z. Then by [6, Theorem 3.2], $\kappa_k(f) = \xi_K^*(f)\,\omega^{-h}$ belongs to $A\,\hat{\otimes}\,k$ for any $f \in T_{g,h}(k)$, and κ_k satisfies the following:

(a) $\kappa_k \colon T_{g,h}(k) \to A\,\hat{\otimes}\,k$ is an injective k-linear homomorphism, and

$$\kappa_k(f \cdot f') = \kappa_k(f) \cdot \kappa_k(f')$$

for Teichmüller modular forms f, f' of degree g over k.

(b) If $\iota \colon k \to k'$ is a homomorphism of fields, then for $f \in T_{g,h}(k)$, $\kappa_{k'}(\iota(f)) = \iota(\kappa_k(f))$.

(c) If k' is a field extension of k and $f \in T_{g,h}(k')$ satisfies that $\kappa_{k'}(f) \in A\,\hat{\otimes}\,k$, then $f \in T_{g,h}(k)$.

(d) Put

$$B = \mathbf{Z}[q_{ij}^{\pm 1}(i, j \in \{1, ..., g\}, i \neq j)][[q_{11}, ..., q_{gg}]]$$

(q_{ij}: variables with symmetry $q_{ij} = q_{ji}$), and for any $\varphi \in S_{g,h}(k)$, let $F(\varphi) \in B\,\hat{\otimes}\,k$ be the arithmetic Fourier expansion of φ with respect to the trivialization by ω^h [1, Chap. V; 3, Section 6; 4, Chap. V]). Then $\kappa_k(\tau^*(\varphi)) = F(\varphi)|_{q_{ij} = p_{ij}}$.

2.4. Elements of A and B are called primitive if these are not congruent to 0 modulo p for any prime p. Evidently, if $u, v \in A$ (or $\in B$) are primitive and equal up to constant in $\mathbf{Q}$, then $u = \pm v$. By the q-expansion principle [4, p. 140], $\varphi \in S_{g,h}(\mathbf{Z})$ satisfies that $\varphi \bmod (p) \neq 0$ in $S_{g,h}(\mathbf{F}_p)$ for any p if and only if its Fourier expansion $F(\varphi)$ is primitive. The following assertion is a slight improvement of a result of Tsuyumine [16], considering the rationality of modular forms.

PROPOSITION 2.5. *There exists a unique integer N_g such that*

$$\kappa_{\mathbf{Q}}(\tau^*(\theta_g))/N_g = (F(\theta_g)/N_g)|_{q_{ij} = p_{ij}}$$

is in A and primitive and that $M\tau^*(\theta_g)/N_g$ has a square root in $T_{g,h(g)}(\mathbf{Q})$ for some positive integer M.

Proof. Let $\mathcal{D}$ be the divisor of $\mathcal{M}_g \otimes \bar{\mathbf{Q}}$ consisting of curves C which have an invertible sheaf L satisfying that $L^{\otimes 2} \cong \Omega_C$ and that the dimension of $\Gamma(C, L)$ is positive and even. Then as is shown in [16, the proof of Theorem 1], $\mathcal{D}^2$ is the divisor of $\tau^*(\theta_g)$, and hence a Teichmüller modular form of weight $h(g)$ with divisor $\mathscr{D}$, which exists and is uniquely determined up to constant by [14, Theorem 2], is a root of $\tau^*(\theta_g)$ up to constant. Since $\mathscr{D}$ is defined over $\mathbf{Q}$, a root of $\tau^*(\theta_g)$ times some number can be defined over $\mathbf{Q}$, and hence we can take N_g as above.

COROLLARY 2.6. *The power series $M(F(\theta_g)/N_g)|_{q_{ij}=p_{ij}}$ has a square root in $A \hat{\otimes} \mathbf{Q}$.*

3. FOURIER COEFFICIENTS OF THETA CONSTANTS

3.1. Expanding the theta constants by

$$q_{ij}^{1/n} = \exp(2\pi\sqrt{-1}\, z_{ij}/n) \qquad (n \in \mathbf{N},\, Z = (z_{ij})_{1 \leqslant i,\, j \leqslant g} \in H_g),$$

we have

$$\theta_g \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} \in \mathbf{Z}[q_{ij}^{\pm 1/4}(i \neq j)][[q_{11}^{1/8}, ..., q_{gg}^{1/8}]]$$

and $\theta_g = F(\theta_g) \in B$ (in what follows, we do not distinguish between Siegel modular forms and their Fourier expansions). We will determine their minimal degree terms with respect to $q_{11}^{1/8}, ..., q_{gg}^{1/8}$. For a vector $\mathbf{v} = (v_1, ..., v_g) \in \mathbf{R}^g$, let

$$|\mathbf{v}| = \sum_{k=1}^{g} |v_k|$$

be its norm. Then a term in $\theta_g[\begin{smallmatrix}\mathbf{a}\\ \mathbf{b}\end{smallmatrix}]$ takes minimal degree when $|\mathbf{n}+\mathbf{a}|$ is minimal. Hence for $\mathbf{a} = \mathbf{0}$,

$$\theta_g \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix} = 1 + \cdots$$

is primitive, and for $\mathbf{a} \neq \mathbf{0}$, the power series $\frac{1}{2}\theta_g[\begin{smallmatrix}\mathbf{a}\\ \mathbf{b}\end{smallmatrix}]$ has integral coefficients because $\mathbf{n}$ and $-\mathbf{n}-2\mathbf{a}$ ($\mathbf{n} \in \mathbf{Z}^g$) give a same term in the sum.

For each subset S of $\{1, ..., g\}$, let $\mathbf{v}_S = (v_1, ..., v_g)$ be the vector given by

$$v_k = \begin{cases} 1 & (k \in S) \\ 0 & (k \notin S), \end{cases}$$

and for $\mathbf{b} \in \{0, 1/2\}^g$ with $|\mathbf{b}| \in \mathbf{Z}$, let

$$P_{\mathbf{b}} = \frac{1}{2} \sum_{S \subset \{1, ..., g\}} (-1)^{2\mathbf{v}_S^t \mathbf{b}} \prod_{i \in S,\, j \notin S} X_{ij}$$

be a polynomial of variables X_{ij} $(i, j \in \{1, ..., g\}, i \neq j)$ with symmetry $X_{ij} = X_{ji}$. Since S and its complement give a same term in this sum, $P_{\mathbf{b}}$ has integral coefficients.

PROPOSITION 3.2. *For any* $\mathbf{b} \in \{0, 1/2\}^g$ *with* $|\mathbf{b}| \in \mathbf{Z}$, $P_{\mathbf{b}}$ *is a primitive element of* $\mathbf{Z}[X_{ij}]$ *which is prime in* $\mathbf{C}[X_{ij}]$.

Proof. This follows from that $P_{\mathbf{b}}$ is of degree $\leqslant 1$ with respect to each $\{1, ..., g\} - T$, X_{ij}, and that for proper and nonempty subsets S, T of $\{1, ..., g\}$ with $S \neq T$, $\prod_{i \in S, j \notin S} X_{ij}$ is not divisible by $\prod_{i \in T, j \notin T} X_{ij}$.

3.3. Put $\mathbf{i} = (\frac{1}{2}, ..., \frac{1}{2})$. Since

$$\exp[\pi \sqrt{-1}\, (\mathbf{i} - \mathbf{v}_S)\, Z^t(\mathbf{i} - \mathbf{v}_S)] = \prod_{1 \leqslant i, j \leqslant g} q_{ij}^{1/8} \prod_{i \in S, j \notin S} q_{ij}^{-1/4} \prod_{i \notin S, j \in S} q_{ij}^{-1/4}$$
$$= \prod_{1 \leqslant i, j \leqslant g} q_{ij}^{1/8} \prod_{i \in S, j \notin S} q_{ij}^{-1/2},$$

for any $\mathbf{b} \in \{0, \frac{1}{2}\}^g$ with $|\mathbf{b}| = 2\mathbf{i}^t \mathbf{b} \in \mathbf{Z}$,

$$\frac{1}{2}\theta_g \begin{bmatrix} \mathbf{i} \\ \mathbf{b} \end{bmatrix} = \frac{1}{2} \sum_{S \subset \{1, ..., g\}} \exp[\pi \sqrt{-1}\, (\mathbf{i} - \mathbf{v}_S)\, Z^t(\mathbf{i} - \mathbf{v}_S)$$
$$+ 2\pi \sqrt{-1}\, (\mathbf{i} - \mathbf{v}_S)^t\, \mathbf{b}] + \cdots$$
$$= (-1)^{|\mathbf{b}|} \Bigg(\prod_{1 \leqslant i, j \leqslant g} q_{ij}^{1/8} \Bigg) \Big(P_{\mathbf{b}} |_{X_{ij} = q_{ij}^{-1/2}} + \cdots \Big),$$

and hence this is primitive by Proposition 3.2. When $\mathbf{a} \neq \mathbf{0}$ and $4\mathbf{a}^t\mathbf{b}$ is even, minimal degree terms of $\theta_g[{}^{\mathbf{a}}_{\mathbf{b}}]$ $(\mathbf{a} = (a_k)_{1 \leqslant k \leqslant g}, \mathbf{b} = (b_k)_{1 \leqslant k \leqslant g})$ are equal to those of $\theta_{|I|}[{}^{\mathbf{a}_I}_{\mathbf{b}_I}]$ $(\mathbf{a}_I = (a_k)_{k \in I}, \mathbf{b}_I = (b_k)_{k \in I})$, where $I = \{k = 1, ..., g \mid a_k = \frac{1}{2}\}$ of which cardinal number is denoted by $|I|$, and hence $\frac{1}{2}\theta_g[{}^{\mathbf{a}}_{\mathbf{b}}]$ is primitive. Since there are $2^{g-1}(2^g - 1)(= 4h(g) - 2^g)$ even pairs $(\mathbf{a}, \mathbf{b})$ with $\mathbf{a} \neq \mathbf{0}$, we have

PROPOSITION 3.4. *The Siegel modular form* $2^{2^{g-1}(1-2^g)}\theta_g$ *is defined over* $\mathbf{Z}$ *and is primitive.*

4. THETA AS A TEICHMÜLLER MODULAR FORM

4.1. Let $\psi: \mathbf{C}[X_{ij}^2] \to \mathbf{R} \otimes \mathbf{C}$ be the ring homomorphism over $\mathbf{C}$ obtained by the substitution $X_{ij}^2 = [x_i, x_{-i};\ x_j, x_{-j}]^{-1}$ $(i, j \in \{1, ..., g\}$, $i \neq j)$. Put

$$P = \prod_{|\mathbf{b}| \in \mathbf{Z}} P_{\mathbf{b}} : \text{the product of } 2^{g-1} P_{\mathbf{b}}\text{'s}.$$

Since for each i, j, $P_{\mathbf{b}}|_{X_{ij}=-X_{ij}} = P_{\mathbf{b}^*}$, where

$$\mathbf{b}^* = (b_k^*)_{1 \leqslant k \leqslant g}; \qquad b_k^* = \begin{cases} 1/2 - b_k & (k = i, j) \\ b_k & (k \neq i, j), \end{cases}$$

P is a polynomial of X_{ij}^2. Put

$$\theta_g^* \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = \Bigl(\prod_{i, j \in I} q_{ij}^{-1/8} \Bigr) \theta_g \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix},$$

where $I = \{k = 1, ..., g \mid a_k = \frac{1}{2}\}$. Then as power series of $y_1, ..., y_g$ with coefficients in R, we have

$$\Bigl(\prod_{|\mathbf{b}| \in \mathbf{Z}} \tfrac{1}{2}\, \theta_g^* \begin{bmatrix} \mathbf{i} \\ \mathbf{b} \end{bmatrix} \Bigr) \Bigr|_{q_{ij} = p_{ij}} = \Bigl(\prod_{|\mathbf{b}| \in \mathbf{Z}} (-1)^{|\mathbf{b}|} \Bigr) \psi(P) + \cdots.$$

Proposition 4.2. *P has no root in $\mathbf{C}[X_{ij}]$.*

Proof. This follows from Proposition 3.2.

Proposition 4.3. *Under $x_1 = x_{-2}, x_2 = x_{-3}, ..., x_g = x_{-1}$, $\psi(P)$ becomes* 1. *In particular, $\psi(P)$ is primitive.*

Proof. If $x_1 = x_{-2}, x_2 = x_{-3}, ..., x_g = x_{-1}$, then $X_{12} = X_{23} = \cdots = X_{g1} = 0$, and hence $\prod_{i \in S, j \notin S} X_{ij} = 0$ for $S \neq \emptyset, \{1, ..., g\}$. Therefore, $\psi(P)|_{x_1 = x_{-2}, ..., x_g = x_{-1}} = 1$ which implies that $\psi(P)$ is primitive.

Theorem 4.4. *The integer N_g in Proposition* 2.5 *is determined by*

$$N_g = \begin{cases} -2^{28} & (g = 3) \\ 2^{2^{g-1}(2^g - 1)} & (g \geqslant 4), \end{cases}$$

and $\tau^*(\theta_g)/N_g$ has a root in $T_{g, h(g)}(\mathbf{Q})$.

Proof. Put

$$\theta_g^* = \prod_{2\mathbf{a}^t\mathbf{b} \in \mathbf{Z}} \theta_g^* \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}.$$

Then θ_g/θ_g^* and $(\theta_g/\theta_g^*)|_{q_{ij} = p_{ij}}$ are the squares of primitive elements of B and A, respectively, and hence in the proof, we may replace θ_g by θ_g^*. As seen above, if $\mathbf{b}' = (b_k')_{1 \leqslant k \leqslant g}$ satisfies that $b_k' = b_k$ for any k with $a_k = \frac{1}{2}$, then minimal degree terms of $\theta_g^*[{}^{\mathbf{a}}_{\mathbf{b}}]$ are equal to those of $\theta_g^*[{}^{\mathbf{a}}_{\mathbf{b}'}]$. Hence by 3.3, 4.1, and Proposition 4.3, the sum of minimal degree terms of

$$\left(\prod_{\mathbf{a}\neq\mathbf{0},\,\mathbf{i}} \tfrac{1}{2}\,\theta_g^*\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}\right)\Bigg|_{q_{ij}=p_{ij}}$$

is the square of a primitive element of A. Therefore, using 4.1 and Proposition 4.3, one can see that

$$N_g = 2^{2^{g-1}(2^g-1)} \prod_{|\mathbf{b}|\in\mathbf{Z}} (-1)^{|\mathbf{b}|}$$

and that $\tau^*(\theta_g)/N_g$ has a root in $T_{g,\,h(g)}(\mathbf{Q})$. The number of $\mathbf{b}\in\{0,\frac{1}{2}\}^g$ with $|\mathbf{b}|\in\mathbf{Z}-2\mathbf{Z}$ is

$$\begin{aligned}\sum_{k=0}^{[(g-2)/4]}\binom{g}{4k+2} &= \tfrac{1}{4}\{2^g-(1+\sqrt{-1})^g-(1-\sqrt{-1})^g\}\\ &= \tfrac{1}{4}(2^g-2(\sqrt{2})^g\cos(g\pi/4))\end{aligned}$$

which is odd only if $g=3$.

THEOREM 4.5. *$\psi(P)$ has a root as a primitive element of R.*

Proof. By Corollary 2.6 and the proof of Theorem 4.4, the sum $\psi(P)$ of minimal degree terms of

$$\left(\prod_{|\mathbf{b}|\in\mathbf{Z}} (-1)^{|\mathbf{b}|}\,\tfrac{1}{2}\,\theta_g^*\begin{bmatrix}\mathbf{i}\\ \mathbf{b}\end{bmatrix}\right)\Bigg|_{q_{ij}=p_{ij}}$$

has a root in $A\hat{\otimes}\mathbf{Q}$, and hence $\sqrt{\psi(P)}\in R\otimes\mathbf{Q}$. Since R is normal and $\psi(P)$ is primitive, $\sqrt{\psi(P)}$ is a primitive element of R.

EXAMPLE 4.6. In the case when $g=3$,

$$\begin{aligned}P &= (1+X_{12}X_{23}+X_{23}X_{31}+X_{31}X_{12})(1+X_{12}X_{23}-X_{23}X_{31}-X_{31}X_{12})\\ &\quad\times(1-X_{12}X_{23}+X_{23}X_{31}-X_{31}X_{12})(1-X_{12}X_{23}-X_{23}X_{31}+X_{31}X_{12}),\end{aligned}$$

and hence we have

$$\begin{aligned}\sqrt{\psi(P)} &= \pm\frac{\prod_{k=1}^{3}(x_k-x_{-k})}{\prod_{1\leqslant i<j\leqslant 3}\{(x_i-x_j)(x_{-i}-x_{-j})\}}\\ &\quad\times\{(x_1-x_{-2})(x_2-x_{-3})(x_3-x_{-1})\\ &\quad-(x_1-x_{-3})(x_3-x_{-2})(x_2-x_{-1})\}.\end{aligned}$$

PROPOSITION 4.7. *If* $g=3$, *then* $\sqrt{\psi(P)} \notin \mathrm{Im}(\psi)$. *Consequently*, f_3 *cannot be induced from Siegel modular forms.*

Proof. Since

$$\{(Y_{ij}=[x_i, x_{-i}; x_j, x_{-j}]^{-1})_{1\leqslant i<j\leqslant 3} \mid x_{\pm 1}, x_{\pm 2}, x_{\pm 3} \in \mathbf{C}, x_k \neq x_l (k\neq l)\}$$

is Zariski dense in $\mathbf{A}^3_{\mathbf{C}}=\{(Y_{ij})_{1\leqslant i<j\leqslant 3}\}$, if there were $H\in\psi^{-1}(\sqrt{\psi(P)})$, then we have $H^2=P$ which contradicts Proposition 4.2.

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